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(Residential Autonomous College under University of Calcutta)

B.A./B.SC. FIFTH SEMESTER EXAMINATION, DECEMBER 2012

THIRD YEAR

MATHEMATICS (Honours)

Date : 21/12/2012

Time : 11 am – 3 pm

Paper : VI

Full Marks : 100

(*Use separate answer-book for each group*)

Group – A

Unit-I

*Answer **any six** questions* [6x5=30]

1. a) Let $f=\lambda u,\ g=\lambda v$ where *f, g* are homogeneous functions of degree *n* and λ be homogeneous function of degree *m*. Prove that if all concerned partial derivatives are continuous, then $\dfrac{\partial(f,g)}{\partial(x,y)}=\dfrac{n\lambda^2}{n-m}\cdot\dfrac{\partial(u,v)}{\partial(x,y)}$ where $n\neq m$. [3]

   b) Examine the existence of Implicit function near the point indicated:
   $$\sin(xy)-e^{xy}-x^2y+1=0,\ (1,0)$$ [2]

2. If $f(x,y)=x^2\tan^{-1}\left(\dfrac{y}{x}\right)-y^2\tan^{-1}\left(\dfrac{x}{y}\right),\ xy\neq 0,\ f(x,0)=0=f(0,y)$.

   Show that $f_{xy}(0,0)\neq f_{yx}(0,0)$. Which condition(s) of Schwarz's theorem is/are not satisfied by *f* ? [2+3]

3. Assuming *z* as a twice differentiable function of *x* and *y*, transform the equation
   $$\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0$$
   Taking $u=x+y,\ v=x-y$ for the new independent variables and $w=xy-z$ for the new function $w\equiv w(u,v)$. [5]

4. Let *S* be an open subset of $\mathbb{R}^n$ and assume that $f:S\to\mathbb{R}^m$ be differentiable at each point of *S*. Let *x* and *y* be two points in *S* such that the line joining them $L(x,y)\subset S$. Prove that for every vector '*a*' in $\mathbb{R}^m$, there is a point ξ in $L(x,y)$ such that
   $$a\cdot\{f(y)-f(x)\}=\left(a\cdot\{(y-x)f'(\xi)\}\right).$$ [5]

5. Let *f* be continuously differentiable on an open set $E\subset\mathbb{R}^n$. Choose a unit vector $\underset{\sim}{u}=\alpha_1\underset{\sim}{u}_1+\cdots+\alpha_n\underset{\sim}{u}_n$ and a point $\underset{\sim}{x}_0\in E$.

   Put $g(t)=f(\underset{\sim}{x}_0+t\underset{\sim}{u})\ (-\delta<t<\delta)$ where δ is chosen so that $\underset{\sim}{x}_0+t\underset{\sim}{u}\in E$ if $|t|<\delta$. Prove that g' is continuous on $(-\delta,\delta)$, and $g'(t)=\sum_{i=1}^{n}\alpha_i D_i f(\underset{\sim}{x})\cdot(\underset{\sim}{x}=\underset{\sim}{x}_0+t\underset{\sim}{u})$. [5]

6. If $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=1$, show that a stationary value of $a^3x^2+b^3y^2+c^3z^2$ is given by $ax=by=cz$ and this gives an extreme value if $abc(a+b+c)$ is positive. [5]

7. a) Let $u=f(x,y,z),\ v=g(x,y,z)$ have continuous first order partial derivatives in a region *B* and let they satisfy a functional relation $F(u,v)=0$. Prove that all two-rowed determinants of the Jacobian matrix $J=\begin{pmatrix}u_x & u_y & u_z\\ v_x & v_y & v_z\end{pmatrix}$ must vanish. [3]

b) Is f continuous at (0, 0)?

$$f(x,y)=\begin{cases}0, & xy\neq 0\\ 1, & xy=0\end{cases}.$$ [2]

8. Let $f(x,y)=(|x+y|+x+y)^k,(x,y)\in\mathbb{R}^2$. Find the values of k for which f is differentiable at each point of $\mathbb{R}^2$. [5]

9. Let (x, y) approach (0, 0) along $y = -x$. Find $\lim \dfrac{\sin xy + xe^x - y}{x\cos y+\sin 2y}$. [5]

Unit-II

Answer <u>any three</u> questions from Q. No. 10 to Q. No. 14 and <u>any one</u> from Q. No. 15 to Q. No. 16 [3x5+1x5]

10. Prove that the centres of spheres which touch the straight lines $y = mx, z = c$ and $y = -mx, z = -c$ lie on the surface $mxy+cz(1+m^2)=0$. [5]

11. Prove that the locus of the feet of the perpendiculars drawn from the centre of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ upon the tangent planes is $a^2x^2+b^2y^2+c^2z^2=(x^2+y^2+z^2)^2$. [5]

12. The enveloping cone of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ is cut by the plane $z = 0$ in a parabola. Show that its vertex lies on the planes $z = \pm c$. [5]

13. Show that the perpendiculars from the origin to the generators of the hyperboloid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ lie upon the cone $\dfrac{a^2(b^2+c^2)^2}{x^2}+\dfrac{b^2(c^2+a^2)^2}{y^2}=\dfrac{c^2(a^2-b^2)^2}{z^2}$. [5]

14. Reduce the equation $11x^2+10y^2+6z^2+4zx-8yz-12xy+72x-72y+36z+150=0$ to its canonical form and state the nature of the conicoid represented by it. [4+1]

15. Verify Green's theorem in the plane for $\int_C (3x^2-8y^2)dx+(4y-6xy)dy$ where C is the boundary of the region defined by $y=\sqrt{x}$ and $y=x^2$. [5]

16. If $\vec{f}=y\hat{i}+(x-2xz)\hat{j}-xy\hat{k}$, evaluate $\int_S (\vec{\nabla}\times\vec{f})\cdot\hat{n}\, dS$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane. [5]

Group – B

(*Answer <u>**any two**</u> questions from Q. No. 17 to Q. No. 19*)

17. a) If A, B, C and D, E, F represent the moments and products of inertia of a rigid body about the coordinate axes respectively, then find the moment of inertia of the body about a line having direction cosines l, m, n w.r.t. the same axes.
 b) A solid homogeneous cone of height h and semi-vertical angle α oscillates about a diameter of its base. Show that the length of the simple equivalent pendulum is $\dfrac{h}{5}(2+3\tan^2\alpha)$.
 c) Define the term "momental ellipsoid at a point". [6+6+3]

18. a) Prove that the kinetic energy of a rigid body moving in two dimensions is the sum of two kinetic energies, one due to translation and the other due to rotation.

b) If a sphere be projected up an inclined plane making an angle α with the horizon, with velocity V and an initial angular velocity Ω (in the direction in which it would roll up) and if $V > a\Omega$, then show that the friction acts downwards at first and upwards afterwards, and prove that the whole time during which the sphere rises is $\frac{17V+4a\Omega}{18g\sin\alpha}$, 'a' being the radius of the sphere and $\frac{1}{7}\tan\alpha$ the coefficient of friction of the plane. [7+8]

19. a) Two equal uniform rods AB and AC are freely jointed at A. They are placed on a table so as to be at right angles. The rod AC is struck by a blow at C in a direction perpendicular to AC. Show that the resulting velocities of the middle points of AB and AC are in the ratio 2 : 7.

b) A plank of mass M is initially at rest along a straight line of greatest slope of a smooth plane inclined at an angle α to the horizon and a man of mass M′ starting from the upper end walks down the plank so that it does not move. Show that he gets to the other end in time $\sqrt{\frac{2M'a}{(M+M')g\sin\alpha}}$, where 'a' is the length of the plank. [8+7]

(*Answer* ***<u>any two</u>*** *questions from Q. No. 20 to Q No. 22*)

20. Find the time of description of an arc of an elliptic orbit for a particle moving under the law of inverse square. [7]

21. A particle slides from a cusp down the arc of a rough cycloid, the axis of which is vertical. Prove that its velocity at the vertex will bear to the velocity at the same point, when the cycloid is smooth, the ratio of $(e^{-\pi\mu}-\mu^2)^{1/2} : (1+\mu^2)^{1/2}$, where μ is the coefficient of friction. [7]

22. A spherical raindrop of radius a cms. falls from rest through a vertical height h, receiving throughout the motion an accumulation of condensed vapour at the rate of K grammes per square cm per second, no vertical motion but gravity acting. Show that when it reaches the ground, its radius will be $K\sqrt{\frac{2h}{g}}\left[1+\sqrt{1+\frac{ga^2}{2hK^2}}\right]$ [7]

(*Answer* ***<u>any one</u>*** *question from Q. No. 23 to Q. No. 24*)

23. When a periodic comet is at its greatest distance from the Sun, its velocity v is increased by a small quantity δv. Show that the comet's least distance from the Sun is increased by $4\left\{\frac{a^3(1-e)}{\mu(1+e)}\right\}^{1/2}.\delta v$ [6]

24. A heavy particle hangs from a point O by a light inelastic string of length 'a'. It is projected horizontally with a velocity v such that $v^2=(2+\sqrt{3})ag$. Show that the string becomes slack when it has described an angle $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$. [6]